6.1 : The Laplace Transform

ENTRY TASK INTEGRATION REVIEW

1. $\int_0^\infty e^{-4t}\,dt = \qquad = \frac{1}{4}$

2. $\int_0^\infty t e^{-3t}\,dt = \qquad = \frac{1}{9}$

CHALLENGE FOR LATER

3. $\int_0^\infty e^{-2t}\cos(t)\,dt = \qquad = \frac{2}{5} = \frac{2}{2^2+1^2}$

GIVE HANDOUT ON BY-PARTS

Def'n Given $f(t)$ that is piecewise continuous (and at most exponential as $t \to \infty$) we define

VARIABLE s

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Ex

1. $\mathcal{L}\{0\} = \int_0^\infty e^{-st} \cdot 0\,dt = 0$

2. $\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} e^{3t}\,dt = \int_0^\infty e^{(3-s)t}\,dt$

$$= \lim_{A\to\infty} \int_0^A e^{(3-s)t}\,dt = \lim_{A\to\infty}\left[\frac{1}{3-s} e^{(3-s)t}\Big|_0^A\right]$$

$s > 3$

$$= \lim_{A\to\infty}\left[\cancel{\frac{1}{3-s} e^{(3-s)A}} - \frac{1}{3-s}\right] = \frac{-1}{3-s}$$

$$= \frac{1}{s-3}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$\boxed{\mathcal{L}\{e^{0t}\} = \mathcal{L}\{1\} = 1/s}$$

3 $\mathcal{L}\{\cos(2t)\} = \int_0^\infty \overset{\alpha}{e^{-st}} \cos(\overset{\beta}{2t})\,dt$

$= \lim_{A\to\infty}\left[\frac{-(-s)}{s^2+2^2} e^{-st}\cos(2t) + \frac{2}{s^2+2^2} e^{-st}\sin(2t)\Big|_0^A\right]$

$= \frac{s}{s^2+2^2}$

ALSO $\boxed{\begin{aligned}\mathcal{L}\{\cos(bt)\} &= \frac{s}{s^2+b^2},\\ \mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2+b^2}\end{aligned}}$

USEFUL

4 TWO GENERAL FACTS

Linearity

$\boxed{\mathcal{L}\{c_1 y_1(t) + c_2 y_2(t)\} = c_1\mathcal{L}\{y_1(t)\} + c_2\mathcal{L}\{y_2(t)\}}$

because $\int_0^\infty e^{-st}(c_1 y_1(t) + c_2 y_2(t))\,dt = c_1\int_0^\infty e^{-st} y_1(t)\,dt + c_2\int_0^\infty e^{-st} y_2(t)\,dt$

$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$
$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$\boxed{\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)}$

$\boxed{\begin{aligned}\mathcal{L}\{e^{at}\cos(bt)\}(s) &= \mathcal{L}\{\cos(bt)\}(s-a) = \frac{s-a}{(s-a)^2+b^2}\\ \mathcal{L}\{e^{at}\sin(bt)\}(s) &= \mathcal{L}\{\sin(bt)\}(s-a) = \frac{b}{(s-a)^2+b^2}\end{aligned}}$

SOME MOTIVATION

CONSIDER

$$y'' - y' - 2y = 0 \Rightarrow r^2 - r - 2 = 0$$
$$y(0) = 1 \qquad (r-2)(r+1) = 0$$
$$y'(0) = 0 \qquad r_1 = 2, \; r_2 = -1$$

$$y(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 + c_2 = 1 \\ y'(0) = 0 \Rightarrow 2c_1 - c_2 = 0 \end{array}\right\} \begin{array}{l} c_1 = 1/3 \\ c_2 = 2/3 \end{array}$$

WE WILL LEARN TO QUICKLY TURN

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

INTO $\quad \}$ NEXT CLASS

$$\frac{s-1}{(s-2)(s+1)}$$

(AND IT WILL INCLUDE INITIAL CONDITIONS!)

$$y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

$$\updownarrow$$

$$\mathcal{L}\{y(t)\} = \frac{1}{3}\mathcal{L}\{e^{2t}\} + \frac{2}{3}\mathcal{L}\{e^{-t}\}$$
$$= \frac{1}{3}\frac{1}{(s-2)} + \frac{2}{3}\frac{1}{(s+1)}$$

THEN WE USE PARTIAL FRACTIONS

This method will be faster and cleaner for many important applications (BUT WILL NOT ALWAYS BE BETTER)

MORE TO ADD TO OUR TABLE

POLYNOMIALS

[4] $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

$$= \lim_{A\to\infty}\left[-\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st}\Big|_0^A\right]$$

$$= \frac{1}{s^2}$$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$ $\quad$ $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

AND $\boxed{\mathcal{L}\{t^n e^{at}\}(s) = \mathcal{L}\{t^n\}(s-a) = \frac{n!}{(s-a)^{n+1}}}$

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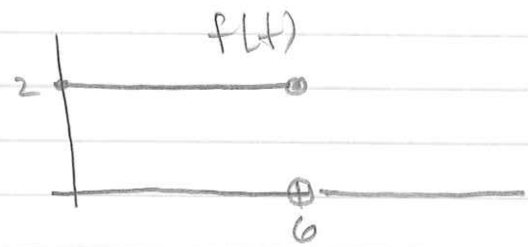
[5] $f(t) = \begin{cases} 2, & 0 \le t \le 6; \\ 0, & t > 6. \end{cases}$



$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ IT IS ZERO PAST 6

$$= \int_0^6 e^{-st}\, 2\, dt$$

$$= -\frac{2}{s} e^{-st}\Big|_0^6 = -\frac{2}{s} e^{-6s} + \frac{2}{s}$$

We can use this
with discontinuous functions!
("ON/OFF SWITCHES")